

Banco de México  
Documentos de Investigación

Banco de México  
Working Papers

N° 2020-03

Many-to-one Matching: Externalities and Stability

Alfredo Salgado  
Banco de México

March 2020

La serie de Documentos de Investigación del Banco de México divulga resultados preliminares de trabajos de investigación económica realizados en el Banco de México con la finalidad de propiciar el intercambio y debate de ideas. El contenido de los Documentos de Investigación, así como las conclusiones que de ellos se derivan, son responsabilidad exclusiva de los autores y no reflejan necesariamente las del Banco de México.

The Working Papers series of Banco de México disseminates preliminary results of economic research conducted at Banco de México in order to promote the exchange and debate of ideas. The views and conclusions presented in the Working Papers are exclusively the responsibility of the authors and do not necessarily reflect those of Banco de México.

# Many-to-one Matching: Externalities and Stability\*

Alfredo Salgado<sup>†</sup>  
Banco de México

**Abstract:** In this paper, we establish sufficient conditions on the domain of preferences and agents' behavior in order to characterize the existence of stable assignments in many-to-one matching problems with externalities. The set of stable matchings depends on what agents believe other agents will do if they deviate. Such sets of reactions are called estimation functions or simply estimations. We show that, unless some restrictions would be imposed on agents' preferences, there is no constraint on agents' behavior that assures the existence of stable matchings. In addition, we introduce a condition on preferences called bottom  $q$ -substitutability that guarantees the existence of at least one stable matching when the set of estimations includes all possible matches. Finally, we analyze a notion of the core and its relation with the set of stable assignments.

**Keywords:** Two-sided matching; Externalities; Stability; Estimation functions; Pessimistic agents; Core.

**JEL Classification:** C71, C78, D62.

**Resumen:** En este artículo, se establecen condiciones suficientes sobre el dominio de las preferencias y el comportamiento de los agentes para caracterizar la existencia de asignaciones estables en problemas de emparejamiento muchos-a-uno con externalidades. El conjunto de emparejamientos estables depende de lo que los agentes creen que haría el resto si se desvían. Dichas reacciones se denominan funciones de estimación o estimaciones. Se demuestra que, a menos que se impongan restricciones sobre las preferencias, no existe ninguna condición sobre el comportamiento de los agentes que asegure la existencia de emparejamientos estables. Además, se introduce una condición sobre las preferencias denominada bottom  $q$ -substitutability que garantiza la existencia de al menos una asignación estable cuando el conjunto de estimaciones incluye todos los emparejamientos posibles. Finalmente, se analiza una noción del núcleo y su relación con el conjunto de asignaciones estables.

**Palabras Clave:** Emparejamientos bilaterales; Externalidades; Estabilidad; Funciones de estimación; Agentes pesimistas; Núcleo.

---

\*Agradezco los consejos de mi director de tesis Antonio Romero Medina y los valiosos comentarios de Aldo Heffner y Alfonso Ceballos que fueron de gran utilidad para mejorar la presentación del artículo.

<sup>†</sup> Dirección General de Investigación Económica. Email: [asalgot@banxico.org.mx](mailto:asalgot@banxico.org.mx).

# 1 Introduction

In standard two-sided matching problems (Gale and Shapley, 1962) such as job markets, college admissions and marriage problems, agents on each side of the market have preferences over the set of agents on the opposite side. In this setting, the presence of externalities is often ignored, since it is usually assumed that agents only care about whom they could be matched with and do not assign any valuation to the match of the others. The absence of externalities simplifies the analysis of this kind of markets and is helpful to prove the existence and establishing several properties of solution concepts such as the set of stable matchings and the core.

However, in several applications agents may not only care about whom they are matched with but also who the other agents' partners are (Dutta and Masso, 1997; Echenique and Yenmez, 2007; Kurucu, 2007; Li, 1993). In these cases, explicitly modelling the presence of externalities could be crucial to understand real-world applications of more general matching markets (Bando, 2012; Hafalir, 2008; Mumcu and Saglam, 2006, 2007, 2010; Roy, 2004; Sasaki and Toda, 1996). Indeed, real-world examples of matching markets with externalities are easy to find. For instance, the presence of couples in a labor market is an important application of two-sided matching problems with externalities since an agent's preferences not only depend on her own employer but also on her partner's employer and potentially others couples' employers. We can also consider some competitive situations where agents interact with each other by teams where complementarity among skills of members are important, such as tournaments and contests. In these cases, competitors care about their own teams and also about the other teams' opponents, since the probability of winning a tournament depends on their own team performance and the competition they face. Other examples can be found in markets with downstream competition where wholesalers care about how retailers are organized to compete with each other or in industries where companies compete in research and development where a firm's outcome not only depends on its own team's research but also on the outcomes for the other companies.

The analysis of matching markets with externalities is challenging for at least two reasons.

First, agents must consider the complete matching configuration between firms and workers in order to define their preferences. Thus, in the presence of externalities, agents' preferences should be defined over the set of all feasible matchings, instead of only over agents on the opposite side of the markets. This issue has crucial implications to analyze the existence of several solution concepts, since some sufficient restrictions over the domain of preferences to guarantee the existence of those solutions may not be well defined in the presence of externalities.

The second issue regards the definition of the solution concept itself for this kind of markets. In standard matching problems, under the well-known solution concept of stability (Gale and Shapley, 1962) agents who plan to block a matching must compare their current and alternative partners in order to evaluate whether deviating from a current match is profitable or not. Under this notion of stability, deviating agents do not care about the reactions of non-deviating agents, since their preferences do not depend on the other agents' partners. In contrast, once externalities are considered, a deviation may be profitable or not depending on the reaction of the rest of the agents. This implies that agents must either anticipate or have some expectations about the reactions of non-deviating agents whenever they plan to block a current matching. According to this argument, it is clear that different assumptions about agents' expected reactions may lead to different notions of stability. For instance, we could assume that, after some coalition of agents deviates, the rest of agents simply remain matched as specified under the current matching, except for the ones who lost their current partners. In contrast, we could assume that after some agents deviate, the rest of players simply break any current match in order to remain with no partners.

In previous literature, matching markets with externalities have been analyzed under two main approaches. In the first case, there is an attempt to establish some assumptions regarding agents' reactions that assure the existence of stable matchings without any restriction over the domain of the agents' preferences. Under the second approach, some restrictions over the agents' preferences that guarantee the existence of stable matching are established under specific assumptions about agents' reactions. Note that, in general, both approaches are not equivalent.

Sasaki and Toda (1996) were the first to analyze the marriage problem with externalities. They propose a notion of stability that is similar to the idea of a conjectural equilibrium<sup>1</sup>. Under this solution concept, agents predict the set of matchings that they consider admissible under a given conjecture about the reactive behaviors of agents. These predictions are called *estimation functions* or simply estimations. Thus for a given set of *estimation functions*, say  $\varphi$ , a matching is  $\varphi$ -stable whenever it is admissible for every agent and not blocked by any man-woman pair or any individual agent. Sasaki and Toda (1996) show that a  $\varphi$ -stable matching may not exist under particular sets of estimation functions. However, they claim that a  $\varphi$ -stable matching exists if and only if all feasible matchings are considered admissible by every agent. In this case, we say that the set of *estimation functions* satisfy a condition called *full admissibility*. Hafalir (2008) extends this model by providing a set of endogenous *estimations* that depends on the agents' preferences. Hafalir's estimations not only guarantee the existence of  $\varphi$ -stable matchings but also shows that the assumption of *full admissibility* is not a necessary condition to assure the existence of  $\varphi$ -stable matchings.

Mumcu and Saglam (2010) also analyze the one-to-one matching problem with externalities. In this case, they propose a notion of stability that satisfies the following two conditions: a) deviating pairs join together while their previous mates, if any, divorce and b) the rest of the agents remain matched as before the deviation. Mumcu and Saglam (2010) show that under this notion of stability stable matchings may not exist. However, they propose restrictions on the agents' preferences that assure the existence of stable assignments. Bando (2012) provides similar results in many-to-one problems with externalities only on the firms' side.

This paper deals with the analysis of many-to-one matching markets with externalities. In particular, we analyze the existence of stable matchings in the sense of Sasaki and Toda (1996). As in previous literature, we find that in this setting a  $\varphi$ -stable matching may not exist. Furthermore, we also show that no set of *estimation functions* (neither endogenous nor exogenous) assures the general existence of  $\varphi$ -stable matchings. This impossibility result contrasts with the case of the marriage problem, where there is at least one set of estimations

---

<sup>1</sup>A conjectural equilibrium is a situation where no agent has incentives to deviate given a conjecture about the reactive behaviors of agents (Rubinstein and Wolinsky, 1994; Azrieli, 2009).

that always guarantees the existence of  $\varphi$ -stable matchings.<sup>2</sup> According to these results, we can analyze our problem in two different ways. On the one hand, we can fix a set of *estimations functions* and find reasonable restrictions on the domain of preferences that lead to have a stable assignment. On the other hand, we can fix a restriction on the domain of preferences and find a reasonable set of *estimations functions* that result in the existence of a stable match.

According to the first approach, we consider a benchmark model where the set of estimation functions satisfies the condition of *full admissibility*. As established in previous literature, it is necessary some kind of substitutability of firms' preferences in order to guarantee the existence of stable assignments in standard two-sided matching problems. Intuitively, this condition implies that complementarities among workers are not strong enough to show increasing returns to workers, i.e. if a worker has been selected from a pool of candidates, he will still be chosen when some of his colleagues are not available anymore (Kelso and Crawford, 1982). Furthermore, this constraint on agents preferences can be easily generalized to more realistic environments with fixed firms' quotas of workers by the condition of *q-substitutability* (Cantala, 2004).

However, it is clear that in the presence of externalities it would be very difficult to impose directly any kind substitutability in the model since agents preferences are defined on the set of all feasible matchings instead of agents on the opposite side of the market. In this setting, potential gains and losses of a match will also depend on how the rest of agents are matched. Nevertheless, under these conditions agents are still able to minimize the worst-case potential losses of a match by using a minimax decision rule that protects them from the risk involved in the decisions of other players.

For instance, in school choice or college admissions, a student's decision to attend a college not only depends on school characteristics itself and the rest of students attending that school but could also depends on the rest students attending other schools. This implies that the value of a match with a given school and a set of partners would depend on the set of matchings that the student considers admissible, i.e. every admissible match among the

---

<sup>2</sup>See, Sasaki and Toda (1996) and Hafalir (2008).

rest of students and schools can be considered as an opponents' best response. Hence, in the absence of additional information that allows students to discriminate between admissible matchings it is reasonable to minimize the risk from others' decisions by choosing the best admissible match among the worst-case potential ones.

According to the previous argument, we propose a constraint over firms' preferences called *bottom  $q$ -substitutability* that generalizes the condition of  *$q$ -substitutability* for many-to-one matching problems with externalities. This condition guarantees the existence of at least one stable assignment under the assumption of *full admissibility*. Intuitively, for any given set of estimation functions, firms and workers use a minimax decision rule over the set of all admissible matchings. This allows agents to compare sets of workers and firms instead of matchings, respectively. The condition of *bottom  $q$ -substitutability* implies that firms would have  *$q$ -substitutable* preferences when they compare sets of workers under this minimax behavior.

Related to the second approach and given that full admissibility is not a necessary condition to solve our problem, we consider a model with *pessimistic* agents in order to rationalize the set of estimation functions.<sup>3</sup> In this setting, we construct a set of endogenous *pessimistic estimation functions* that depends on the agents' preferences. We show that these estimations do not satisfy the condition of full admissibility. Further, we show that under the set of *pessimistic estimations*, a  $\varphi$ -stable matching exists provided preferences are *bottom  $q$ -substitutable*.

The last part of the paper deals with the analysis of the core in many-to-one matching markets with externalities. Sasaki and Toda (1996) introduce a notion of the core in marriage problems with externalities. They show that the core and the set of pair-wise  $\varphi$ -stable matchings do not coincide. Further, the core may be empty for some instances of the problem. We propose an alternative notion of the core that depends on the set of estimation functions called the  $\varphi$ -core. Our main result shows that for any set of *estimation functions* the set of  $\varphi$ -stable matchings and the  $\varphi$ -core always coincide. This result contrasts with previous findings and

---

<sup>3</sup>Sasaki and Toda (1996) show that a  $\varphi$ -stable matching may not exist when only one of the agents is not pessimistic enough.

implies that all properties of the set of stable  $\varphi$ -matchings naturally extend to the  $\varphi$ -core.

The rest of the paper is organized as follows. In Section 2, we introduce the model and some basic examples; in Section 3, we introduce the condition of *bottom  $q$ -substitutability* in order to characterize the existence of  $\varphi$ -stable matchings; in Section 4, we introduce the set of *pessimistic estimations functions*; in Section 5, we introduce the  $\varphi$ -core; in Section 6, we present some conclusions. All proofs are in the Appendix.

## 2 The Model

Let  $F$  denote the set of firms and let  $W$  denote the set of workers.  $F$  and  $W$  are disjoint and finite sets with  $m \geq 1$  and  $n \geq 1$  members, respectively. Each worker  $w \in W$  wants to work for at most one firm while each firm  $f \in F$  has a quota  $q_f \leq n$  that denotes the maximum number of workers that the firm is able to hire. We denote by  $H_f = \{S \in 2^W : |S| \leq q_f\}$  the set of all subsets of workers (including the empty set of workers) that a firm  $f$  is able to hire. A matching is a rule that specifies a group of workers for each firm and a firm for each worker allowing for the possibility that some agents remain unmatched. Formally,

**Definition 1** A matching is a mapping  $\mu : F \cup W \rightarrow 2^{F \cup W}$  such that:

1.  $|\mu(w)| = 1$  for all  $w \in W$  and either  $\mu(w) \cap F \neq \emptyset$  or  $\mu(w) = \{w\}$ ;
2.  $\mu(f) \in H_f$  for all  $f \in F$ . If  $\mu(f) = \emptyset$  then the firm  $f$  does not hire any worker; and
3.  $\mu(w) = \{f\}$  if and only if  $w \in \mu(f)$ .

Let  $\mathcal{M}$  denote the set of all feasible matchings given  $F$ ,  $W$  and  $q$ . In standard matching problems, agents have ordinal preferences over the set of agents on the opposite side of the market, i.e. workers have preferences over the set of firms and the prospect of remaining unmatched, while firms have preferences over the set of groups of workers including the empty set. This specification of preferences leads agents to only care about whom they could be matched with and not the other agents' partners, i.e. the presence of externalities is not considered. In this paper, we consider a more general setting where agents' valuations



over potential partners may depend on the complete matching configuration between firms and workers. This specification of agents' preferences explicitly introduces the presence of externalities in many-to-one matching problems.

In this setting, agents' preferences would be defined by a preference relation over the set of all feasible assignments. Formally, each agent  $a \in F \cup W$  has a complete, strict and transitive preference relation over the set of all feasible matchings  $\mathcal{M}$  denoted by  $P_a^*$ . Thus for any given two pair of feasible matchings  $\mu, \mu' \in \mathcal{M}$ , the preference relation  $\mu P_a^* \mu'$  means that the agent  $a \in F \cup W$  prefers the assignment  $\mu(a)$  under the matching  $\mu$  to the assignment  $\mu'(a)$  under the matching  $\mu'$ . Note that this preferences are even more general than a simple comparison between agents (sets of agents) on the opposite side of the market, since an agent  $a \in F \cup W$  may have the same assignment under two different matchings  $\mu$  and  $\mu'$  (i.e.  $\mu(a) = \mu'(a)$ ) without being indifferent between them. According to this preferences, an agent would be indifferent between two matchings only if these matchings are identical. Formally, two matchings  $\mu, \mu' \in \mathcal{M}$  are identical  $\mu = \mu'$  if and only if  $\mu(a) = \mu'(a)$  for all  $a \in F \cup W$ . For each agent  $a \in F \cup W$ , let  $R_a^*$  denote the weak preference relation induced by  $P_a^*$ , so that for any two feasible matchings  $\mu, \mu' \in \mathcal{M}$ , the preference relation  $\mu R_a^* \mu'$  means either  $\mu P_a^* \mu'$  or  $\mu = \mu'$ . Let  $P^* = (P_{f_1}^*, \dots, P_{f_m}^*; P_{w_1}^*, \dots, P_{w_n}^*)$  denote the profile of agents' preferences, thus a matching problem with externalities is a four-tuple  $(F, W, P^*, q)$ .

We consider a solution concept for matching markets with externalities based on the concept of estimation functions (Sasaki and Toda, 1996; Hafalir, 2008). According to this solution concept, in the presence of externalities agents form expectations about the set of matching that they consider admissible. Agents use such expectations in order to evaluate whether deviating from a current matching is profitable or not. These expectations are called *estimation functions* (or simply *estimations*).

Before introducing a formal definition of the set of estimation functions, we require some additional notation. Let  $A(f, S) = \{\mu \in \mathcal{M} : \mu(f) = S \in H_f\}$  denote the set of all feasible matchings where the firm  $f$  and the feasible set of workers  $S$  are matched. In a similar way, let  $A(w, a)$  denote the set of all matchings where the worker  $w \in W$  and the agent  $a \in F \cup \{w\}$  are matched. For each firm  $f \in F$  and any set of feasible workers  $S \in H_f$ , a es-

*estimation function*  $\varphi_f$  specifies a non-empty subset of matchings where  $f$  and  $S$  are matched. Formally, for each  $f \in F$ ,  $\varphi_f$  maps a non-empty subset of matchings  $\varphi_f(S) \subset A(f, S)$  for every  $S \in H_f$ . In a similar way, for each worker  $w \in W$  the *estimation function*  $\varphi_w$  maps a non-empty subset of matchings  $\varphi_w(a) \subset A(w, a)$  for every  $a \in F \cup W$ . Let  $\varphi = \{(\varphi_f(\cdot), \varphi_w(\cdot)) : f \in F \text{ and } w \in W\}$  denote a set of *estimation functions* of a matching problem with externalities  $(F, W, P^*, q)$ . Note that a set of *estimation functions* may be either exogenously determined or an endogenous mapping which could depend on agents' preferences and other additional conditions.

As we argued before, in the presence of externalities whenever some agents plan to block a matching they must either anticipate or have some expectations about the reactions of non-deviating ones. Since in this setting a profitable deviation depends on those expectation, our notion of stability must include the concept of estimation functions. According to the previous argument, a first requirement for a stable matching is to be admissible for all agents. Intuitively, a stable matching has to be consistent with the fact that agents only care about matchings that they consider admissible. Hence, we introduce the following definition,

**Definition 2** *Given a set of estimation functions  $\varphi$ , a matching  $\mu$  is  $\varphi$ -admissible whenever  $\mu \in \varphi_a(\mu(a))$  for all  $a \in F \cup W$ .*

A second requirement for a stable matching is to have no coalitions of agents that would like to deviate from their current assignment. In order to be consistent with the presence of externalities, agents must consider their estimation functions when they plan to deviate from a current matching. Intuitively, an agent is willing to block a matching only if he will be better off under all admissible matchings after deviating. According to this intuitive argument, we consider the following notion of deviating coalitions.

**Definition 3** *An individual worker  $w \in W$ , such that  $\mu(w) \neq w$ , blocks the matching  $\mu$  if  $\mu' P_w^* \mu$  for all  $\mu' \in \varphi_w(w)$ .*

**Definition 4** *A coalition firm-set of workers  $\{f, S\}$  such that  $S \in H_f$  and  $\mu(f) \neq S$ , blocks the matching  $\mu$  if:*

1.  $\mu' P_f^* \mu$  for all  $\mu' \in \varphi_f(S)$  and
2.  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w(f)$  and all  $w \in S$ .

Then, given a set of *estimations functions*  $\varphi$ , a matching  $\mu$  is  $\varphi$ -stable if it is  $\varphi$ -admissible and not blocked by any individual worker or coalition. Let  $\mathcal{E}_\varphi(F, W, P^*, q)$  denote the set of  $\varphi$ -stable matchings of the problem.

## 2.1 Preliminary Examples

In this section, we analyze some simple examples in order to introduce two main issues that our model faces. First, we show that in general a  $\varphi$ -stable matching may not exist in matching problems with externalities. Second, we provide a crucial result that establishes that no *estimation functions* can guarantee the existence of  $\varphi$ -stable matchings.

In order to simplify, in these examples, we consider a particular set of *estimation functions* according to which every matching is considered admissible for every agent; as we said before, this situation is called *full admissibility*. Formally, we say that a set of *estimation functions*  $\varphi$  satisfies the condition of *full admissibility* if for each firm  $f \in F$ ,  $\varphi_f(S) = A(f, S)$  for all  $S \in H_f$  and for each worker  $w \in W$ ,  $\varphi_w(a) = A(w, a)$  for all  $a \in F \cup \{w\}$ . Let  $\mathcal{E}(F, W, P^*, q)$  denote the set of  $\varphi$ -stable matchings under the conditions of *full admissibility*.

According to our notion of stability, in matching problems with externalities every agent who plans to deviate from a current matching should consider as admissible the set of all feasible matchings after deviating. In contrast, when there are no externalities, agents only consider their current and posterior partners in order to evaluate whether deviating from a current match is profitable or not. Thus, it seems to be more difficult to block a matching in the presence of externalities, since every deviating agent should be better off under all admissible matchings after deviating. This intuitive argument implies that it should be relatively easier to sustain the existence of stable matching in this setting than in problems without external effects. However, it is easy to find examples of matching problems with externalities with no  $\varphi$ -stable matchings, as we show in the following example.

**Example 1** Consider a matching problem with three workers  $W = \{w_1, w_2, w_3\}$  and two firms  $F = \{f_1, f_2\}$  with quotas  $q_{f_1} = 2$  and  $q_{f_2} = 1$ . In order to simplify, we describe a matching by a list of workers' partners in the order  $w_1, w_2, \dots, w_n$ . For instance, the list  $\mu_1 = f_1, f_1, f_2$  means that agents are matched such that  $\mu_1(w_1) = f_1$ ,  $\mu_1(w_2) = f_2$  and  $\mu_1(w_3) = f_2$ . The set of all feasible matchings of this problem is presented in the following table,

Table 1: Set of feasible matchings

$\mu_1 = f_1, f_1, f_2$	$\mu_2 = f_1, w_2, f_2$	$\mu_3 = w_1, f_1, f_2$	$\mu_4 = w_1, w_2, f_2$
$\mu_5 = f_1, f_1, w_3$	$\mu_6 = f_1, w_2, w_3$	$\mu_7 = w_1, f_1, w_3$	$\mu_8 = f_1, f_2, f_1$
$\mu_9 = w_1, f_2, f_1$	$\mu_{10} = f_1, f_2, w_3$	$\mu_{11} = w_1, f_2, w_3$	$\mu_{12} = f_1, w_2, f_1$
$\mu_{13} = w_1, w_2, f_1$	$\mu_{14} = f_2, f_1, f_1$	$\mu_{15} = f_2, f_1, w_3$	$\mu_{16} = f_2, w_2, f_1$
$\mu_{17} = f_2, w_2, w_3$	$\mu_{18} = w_1, f_1, f_1$	$\mu_{19} = w_1, w_2, w_3$	

Source: Own elaboration.

Consider the following agents' preferences over the set of all feasible matchings.

$$P_{f_1}^* = \mu_1, \mu_5, \mu_4, \mu_{11}, \mu_{17}, \mu_{19}, \mu_{10}, \mu_2, \mu_6, \mu_7, \mu_3, \mu_{15}, \mu_{13}, \mu_{16}, \mu_9, \mu_{14}, \mu_{18}, \mu_8, \mu_{12}.$$

$$P_{f_2}^* = \mu_2, \mu_3, \mu_4, \mu_1, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{17}, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_5, \mu_6, \mu_7, \mu_{12}, \mu_{13}, \mu_{18}, \mu_{19}.$$

$$P_{w_1}^* = \mu_1, \mu_2, \mu_5, \mu_6, \mu_8, \mu_{10}, \mu_{12}, \mu_{17}, \mu_{16}, \mu_{15}, \mu_{14}, \mu_3, \mu_4, \mu_7, \mu_9, \mu_{11}, \mu_{13}, \mu_{18}, \mu_{19}.$$

$$P_{w_2}^* = \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{18}, \mu_{15}, \mu_{14}, \mu_7, \mu_5, \mu_3, \mu_1, \mu_{19}, \mu_{17}, \mu_{16}, \mu_{13}, \mu_{12}, \mu_6, \mu_4, \mu_2.$$

$$P_{w_3}^* = \mu_8, \mu_9, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{16}, \mu_{18}, \mu_5, \mu_6, \mu_7, \mu_{10}, \mu_{11}, \mu_{15}, \mu_{19}, \mu_{17}, \mu_1, \mu_2, \mu_3, \mu_4.$$

Now consider a set of estimation functions that satisfies the condition of *full admissibility*. Hence, every matching is  $\varphi$ -admissible and, as a consequence, a candidate to be  $\varphi$ -stable. In order to show that a matching  $\mu$  is not  $\varphi$ -stable, we need to find either a worker or a coalition willing to block that current match  $\mu$ . Consider for instance the case of the matching  $\mu_8 = f_1, f_2, f_1$  and the coalition formed by the firm  $f_1$  and the empty set of workers. It is easy to show that the set of matchings where the firm  $f_1$  and the empty set of workers are matched is  $A(f_1, \emptyset) = \{\mu_4, \mu_{11}, \mu_{17}, \mu_{19}\}$ . Note that by assumption, every matching in the set  $A(f_1, \emptyset)$  is admissible for the firm  $f_1$ . Further, according to  $f_1$ 's preferences it is easy to

observe that  $\mu' P_{f_1}^* \mu_8$  for all  $\mu' \in A(f_1, \emptyset)$ . Hence, the matching  $\mu_8$  cannot be  $\varphi$ -stable, since it is blocked by the coalition  $\{f_1, \emptyset\}$ . By a similar argument, it is easy to show that every feasible matching of this example can be blocked by at least one coalition or worker.

Table 2: **Blocking coalitions**

Matching:	Blocked by:
$\mu_1, \mu_2, \mu_3, \mu_4$	$\{w_3\}$
$\mu_5, \mu_6, \mu_7, \mu_{12}$	$\{f_2, \{w_2\}\}$
$\mu_8, \mu_9, \mu_{10}, \mu_{14}, \mu_{15}$	$\{f_1, \emptyset\}$
$\mu_{11}, \mu_{18}$	$\{f_2, \{w_1\}\}$
$\mu_{13}, \mu_{16}, \mu_{17}, \mu_{19}$	$\{f_1, \{w_1, w_2\}\}$

Source: Own elaboration.

This simple example shows that a  $\varphi$ -stable matching may not exist even when the set of estimation functions satisfies the condition of *full admissibility*. This is an interesting implication since, in contrast with the marriage problem with externalities (Sasaki and Toda, 1996), the condition of *full admissibility* is not sufficient to assure the existence of  $\varphi$ -stable assignments in many-to-one matching problems.

It is interesting to note that our notion of stability also allows the analysis of some particular cases of the matching problem where the presence of externalities is only considered over one side of the market. For instance, we can consider a simple setting with externalities on the firms' side, i.e. firms have preferences defined over the set of all feasible matchings while workers have preferences over the set of firms and the prospect of remaining unmatched, as we analyze in the following example.

**Example 2** Consider a matching problem with two firms  $F = \{f_1, f_2\}$  and three workers  $W = \{w_1, w_2, w_3\}$  with quotas  $q_{f_i} = 2$  for  $i = 1, 2$ . The set of all feasible matchings of this problem is presented in the following table,

Table 3: Set of feasible matchings

$\mu_1 = f_1, f_1, f_2$	$\mu_2 = f_1, f_2, f_2$	$\mu_3 = f_1, f_2, f_1$	$\mu_4 = f_1, f_1, w_3$
$\mu_5 = f_1, w_2, w_3$	$\mu_6 = f_1, w_2, f_1$	$\mu_7 = f_1, w_2, f_2$	$\mu_8 = f_1, f_2, w_3$
$\mu_9 = f_2, f_2, f_1$	$\mu_{10} = f_2, f_1, f_1$	$\mu_{11} = f_2, f_1, f_2$	$\mu_{12} = f_2, f_2, w_3$
$\mu_{13} = f_2, w_2, w_3$	$\mu_{14} = f_2, w_2, f_2$	$\mu_{15} = f_2, w_2, f_1$	$\mu_{16} = f_2, f_1, w_3$
$\mu_{17} = w_1, w_2, f_2$	$\mu_{18} = w_1, f_1, w_3$	$\mu_{19} = w_1, f_2, w_3$	$\mu_{20} = w_1, f_1, f_1$
$\mu_{21} = w_1, f_2, f_2$	$\mu_{22} = w_1, f_1, f_2$	$\mu_{23} = w_1, f_2, f_1$	$\mu_{24} = w_1, w_2, f_1$
$\mu_{25} = w_1, w_2, w_3$			

Source: Own elaboration.

Consider the following firms' preferences over the set of all feasible matching:

$$P_{f_1}^* = \mu_6, \mu_3, \mu_4, \mu_1, \mu_{20}, \mu_{10}, \mu_7, \mu_5, \mu_8, \mu_2, \mu_{22}, \mu_{16}, \mu_{18}, \mu_{11}, \mu_{25}, \mu_{19}, \mu_{17}, \mu_{14}, \mu_{13}, \mu_{12}, \mu_{21}, \mu_{23}, \mu_{24}, \mu_{15}, \mu_9.$$

$$P_{f_2}^* = \mu_{14}, \mu_{11}, \mu_{21}, \mu_2, \mu_{12}, \mu_9, \mu_7, \mu_{22}, \mu_{17}, \mu_1, \mu_{13}, \mu_{16}, \mu_{15}, \mu_{10}, \mu_{19}, \mu_8, \mu_{23}, \mu_3, \mu_{25}, \mu_5, \mu_{20}, \mu_6, \mu_{18}, \mu_{24}, \mu_4.$$

And the following workers' preferences over the set of firms and the prospect of remaining unmatched:

$$P_{w_1} = f_2, f_1, w_1.$$

$$P_{w_2} = f_2, f_1, w_2.$$

$$P_{w_3} = f_1, f_2, w_3.$$

As in the previous example, we consider that firms have a set of estimation functions that satisfy *full admissibility* and block a matching according to our notion of stability in the presence of externalities, i.e. they compare the current matching with all admissible matchings after deviating. On the other hand, workers only make a comparison between their current and prospective partners when they plan to deviate from a current matching.

In order to illustrate the problem, consider the case of the matching  $\mu_3 = f_1, f_2, f_1$ . It is easy to show that  $\mu_3$  can be blocked by the coalition  $\{f_2, \{w_1\}\}$ . First of all, note that for the firm  $f_2$  the set of all admissible matchings after deviating is given by  $A(f_2, \{w_1\}) = \{\mu_{10}, \mu_{13}, \mu_{15}, \mu_{16}\}$ . In addition, according to  $f_2$ 's preferences it is easy to observe that

$\mu' P_{f_2}^* \mu_3$  for all  $\mu' \in A(f_2, \{w_1\})$ . On the other hand, note that according to  $w_1$ 's preferences  $f_2 P_{w_1} \mu_3(w_1)$ . Hence, the matching  $\mu_3$  cannot be  $\varphi$ -stable. By a similar argument, it is easy to show that every feasible matching can be blocked by either a coalition or individual worker as we show in the following table.

Table 4: **Blocking coalitions**

Matching	Blocked by
$\mu_1 = f_1, f_1, f_2$	$\{f_2, \{w_1, w_3\}\}$
$\mu_{10} = f_2, f_1, f_1$	$\{f_2, \{w_1, w_2\}\}$
$\mu_{11} = f_2, f_1, f_2$	$\{f_1, \{w_2, w_3\}\}$
$\mu_2 = f_1, f_2, f_2$	$\{f_2, \{w_1, w_3\}\}$
$\mu_9 = f_2, f_2, f_1$	$\{f_1, \emptyset\}$

Source: Own elaboration.

In addition, any matching that leaves  $w_1$  unmatched is blocked by either  $\{f_1, \{w_1\}\}$  or  $\{f_2, \{w_1\}\}$ . Any matching that leaves  $w_2$  unmatched is blocked by either  $\{f_1, \{w_2\}\}$ ,  $\{f_2, \{w_2\}\}$  or  $\{f_2, \{w_2, w_3\}\}$ . Finally, any matching that leaves  $w_3$  unmatched is blocked by  $\{f_2, \{w_1, w_3\}\}$ . Then, as in the previous example, the set of  $\varphi$ -stable matchings of this problem is empty, i.e.  $\mathcal{E}(F, W, P^*, q) = \emptyset$ .

Sasaki and Toda (1996) also show that in the marriage problem with externalities a  $\varphi$ -stable matching may not exist for a given set of *estimation functions*. However, in addition, they also provide a set of estimations that guarantees such existence. Furthermore, they claim that a  $\varphi$ -stable matching exists if and only if the set of *estimation functions* satisfies *full admissibility*. Hafalir (2008) shows that this conjecture is incorrect, since he characterizes a set of endogenous estimations that does not satisfy *full admissibility* but guarantees the existence of  $\varphi$ -stable matchings in the marriage problem with externalities.

Our examples follow this line and show that in many-to-one matching problems with externalities a  $\varphi$ -stable matching may not exist under particular sets of estimation functions. However, our results are even more general, since we also show that in this setting the condition of *full admissibility* is not sufficient to assure the existence of  $\varphi$ -stable matchings. The following result gives even more insights about the scope of these results, since they allow us to establish more general conclusions about the existence of  $\varphi$ -stable assignments in many-

to-one matching problems.

**Lemma 1** *Let  $(F, W, P^*, q)$  be any instance of the matching problem with externalities. If there are no  $\varphi$ -stable matchings under full admissibility, then there are no  $\varphi$ -stable matchings for any possible set of estimation functions.*

Note that Examples 1 and 2 and Lemma 1 imply the following general impossibility result.

**Theorem 1** *In many-to-one matching problems with externalities, no set of estimation functions  $\varphi$  guarantees the existence of  $\varphi$ -stable matchings.*

The previous result has crucial implications for the analysis of matching problems with externalities. On the one hand, our result shows that for any set of estimation functions there exists at least one matching problem with no  $\varphi$ -stable matchings. On the other hand, this result suggests that a restriction on the domain of agents' preferences seems to be necessary in order to guarantee the existence of  $\varphi$ -stable assignments.

### 3 The Existence of $\varphi$ -stable Matchings under *Full Admissibility*

In the previous section, we argued that in many-to-one matching problems with externalities it is impossible to find a set of estimation functions that guarantees the existence of  $\varphi$ -stable matchings. This result suggests that a restriction on the domain of preferences may be necessary to guarantee such existence. The analysis of standard matching problems reaches similar conclusions, since the existence of stable matchings in many-to-one problems can be guaranteed only under restricted domains of preferences (Roth and Sotomayor, 1990).

As established in previous literature, it is necessary to impose some kind of substitutability on firms' preferences in order to guarantee the existence of stable assignments in standard two-sided matching problems. Intuitively, this condition implies that complementarities



among workers are not strong enough to show increasing returns to workers, i.e. if a worker has been selected from a pool of candidates, he will still be chosen when some of his colleagues are not available anymore (Kelso and Crawford, 1982). Furthermore, this constraint on agents' preferences can be easily generalized to more realistic environments with fixed firms' quotas of workers by the condition of *q-substitutability* (Cantala, 2004).

However, it is clear that in the presence of externalities it would be very difficult to impose directly any kind substitutability in the model since agents' preferences are defined on the set of all feasible matchings instead of agents on the opposite side of the market. In this setting, potential gains and losses of a match will also depend on how the rest of agents are matched. Nevertheless, under these conditions agents are still able to minimize the worst-case potential losses of a match by using a minimax decision rule that protects them from the risk involved in the decisions of other players.

According to the previous argument, we propose a constraint over firms' preferences called *bottom q-substitutability* that generalizes the condition of *q-substitutability* for many-to-one matching problems with externalities. This condition guarantees the existence of at least one stable assignment under the assumption of *full admissibility*. Intuitively, for any given set of estimation functions, firms and workers use a minimax decision rule over the set of all admissible matchings. This allows agents to compare sets of workers and firms instead of matchings, respectively. The condition of *bottom q-substitutability* implies that firms would have *q-substitutable* preferences when they compare sets of workers under this minimax behavior.

We also consider an interesting approach to analyze the problem that consists in constructing a *reduced problem* without externalities that allows us to apply standard results of matching problems without external effects to establish the existence of  $\varphi$ -stable matchings.<sup>4</sup>

---

<sup>4</sup>Obviously, this approach is useful if the reduced problem is well defined and when the existence of stable matching in the reduced problem leads us to establish the existence of  $\varphi$ -stable matchings in the original problem with externalities. Examples of this approach can be found in Shapley and Shubik (1969), Sasaki and Toda (1996), Hafalir (2008) and Klaus and Klijn (2005).

### 3.1 The *Reduced Problem*

In this section, we show how to construct a consistent *reduced problem* for any matching problem with externalities. For this purpose, we require some additional notation. Given a set of *estimation functions*  $\varphi$ , for each firm  $f \in F$  (worker  $w \in W$ ) the matching  $\mu_f^S$  ( $\mu_w^a$ ) satisfies the following conditions: a)  $\mu_f^S \in \varphi_f(S)$  ( $\mu_w^a \in \varphi_w(a)$ ) and b)  $\mu' R_f^* \mu_f^S$  for all  $\mu' \in \varphi_f(S)$  ( $\mu'' R_w^* \mu_w^a$  for all  $\mu'' \in \varphi_w(a)$ ). Then  $\mu_f^S$  ( $\mu_w^a$ ) is the least preferred admissible matching where the firm  $f$  (worker  $w$ ) is matched with a feasible set of workers  $S \in H_f$  (with an agent  $a \in F \cup \{w\}$ ). Note that each of these least preferred matchings is well defined, since agents' preferences are strict and complete over the set of all feasible matchings and by definition the *estimations functions* are nonempty subsets of matchings.

Given any matching problem  $(F, W, P^*, q)$  and a set of *estimation functions*  $\varphi$ , we define for each firm  $f \in F$  a preference relation  $P_f^\varphi$  over the set of feasible sets of workers  $H_f$  in the following way: for any two different subsets of workers  $S, S' \in H_f$ , the preference relation satisfies  $SP_f^\varphi S'$  if and only if  $\mu_f^S P_f^* \mu_f^{S'}$ . In a similar way, for each worker  $w \in W$  and any pair of different agents  $a, a' \in F \cup \{w\}$ , the preference relation  $P_w^\varphi$  satisfies  $aP_w^\varphi a'$  if and only if  $\mu_w^a P_w^* \mu_w^{a'}$ .

Since the preference order  $P_a^*$  is complete, strict and transitive for each agent  $a \in F \cup W$ , we know that  $P_a^\varphi$  is well defined in the sense that it is a complete, strict and transitive preference relation over the set of agents on the opposite side of the market. Hence, each matching problem with externalities  $(F, W, P^*, q)$  and a set of estimation functions  $\varphi$  induce a well defined matching problem without externalities denoted by  $(F, W, P^\varphi, q)$ , where  $P^\varphi = (P_{f_1}^\varphi, \dots, P_{f_m}^\varphi; P_{w_1}^\varphi, \dots, P_{w_n}^\varphi)$  is a profile of the agents' preferences. In addition, for each agent  $a \in F \cup W$ , let  $R_a^\varphi$  denote the weak preference relation associated with  $P_a^\varphi$ . In order to illustrate our notation, consider the following example.

**Example 3** Consider the matching problem already introduced in Example 2. Take, for instance, the feasible subsets of workers  $\{w_1, w_3\}$  and  $\{w_1, w_2\}$  for the firm  $f_1$ . It is easy to show that the least preferred matchings associated with these subsets of workers are  $\mu_{f_1}^{\{w_1, w_3\}} = \mu_3$  and  $\mu_{f_1}^{\{w_1, w_2\}} = \mu_1$ . According to  $f_1$ 's preferences  $\mu_3 P_{f_1}^* \mu_1$ , this induces a

preference relation that satisfies  $\{w_1, w_3\} P_{f_1}^\varphi \{w_1, w_2\}$ . It is easy to construct the whole profile of preferences that characterizes the reduced problem of this example:

$$P_{f_1}^\varphi = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \emptyset, \{w_3\}.$$

$$P_{f_2}^\varphi = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \emptyset.$$

We make the same exercise for the matching problem already introduced in Example 1. In this case, we have the following profile of preferences:

$$P_{f_1}^\varphi = \{w_1, w_2\}, \emptyset, \{w_1\}, \{w_2\}, \{w_3\}, \{w_2, w_3\}, \{w_1, w_3\}.$$

$$P_{f_2}^\varphi = \{w_3\}, \{w_1\}, \{w_2\}, \emptyset.$$

$$P_{w_1}^\varphi = f_1, f_2, w_1.$$

$$P_{w_2}^\varphi = f_2, f_1, w_2.$$

$$P_{w_3}^\varphi = f_1, w_3, f_2.$$

Our notion of stability for matching problem with externalities induces a natural notion of stability for problems without externalities. Such notion of stability depends on the following definitions.

**Definition 5** A matching  $\mu$  is blocked by a worker  $w \in W$  if  $w P_w \mu(w)$ .

**Definition 6** A coalition firm-set of workers  $\{f, S\}$  such that  $S \in H_f$  and  $\mu(f) \neq S$ , blocks the matching  $\mu$  if:

1.  $S P_f \mu(f)$  and
2.  $f R_w \mu(w)$  for all  $w \in S$ .

A matching  $\mu$  is stable in the matching problem  $(F, W, P, q)$ , if it is not blocked by any worker or coalition. Let  $\mathcal{E}(F, W, P, q)$  denote the set of stable matchings of the problem.

The following result is important to characterize the existence  $\varphi$ -stable matchings, it basically says that any  $\varphi$ -admissible matching that is stable in the *reduced problem* is also  $\varphi$ -stable in the original problem.

**Proposition 1** *Let  $(F, W, P^*, q)$  be any matching problem with externalities and  $\varphi$  any set of estimation functions. Then any matching  $\mu$  that satisfies: a)  $\mu \in \varphi_a(\mu(a))$  for all  $a \in F \cup W$  and b)  $\mu \in \mathcal{E}(F, W, P^\varphi, q)$  is  $\varphi$ -stable, i.e.  $\mu \in \mathcal{E}_\varphi(F, W, P^*, q)$ .*

The previous result has a crucial implication, since every matching is  $\varphi$ -admissible under *full admissibility* then any stable matching of the *reduced problem* is also  $\varphi$ -stable. We establish this observation as a Corollary of the previous result.

**Corollary 1** *Assume that the set of estimation functions  $\varphi$ , satisfies full admissibility then  $\mathcal{E}(F, W, P^\varphi, q) \subset \mathcal{E}(F, W, P^*, q)$ .*

Note that the converse of the previous result does not necessarily hold, as we show in the following example.

**Example 4** *Consider a matching problem with three workers  $W = \{w_1, w_2, w_3\}$  and two firms  $F = \{f_1, f_2\}$  with quotas  $q_{f_1} = q_{f_2} = 2$ . Firms have the same preferences as in the matching problem presented in Example 2, while workers' preferences are the following,*

$$P_{w_1}^* = \mu_9, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{10}, \mu_1, \mu_8, \mu_7, \mu_4, \mu_5, \mu_6, \mu_3, \mu_2, \mu_{17}, \mu_{23}, \\ \mu_{24}, \mu_{25}, \mu_{22}, \mu_{18}, \mu_{21}, \mu_{19}, \mu_{20}.$$

$$P_{w_2}^* = \mu_{23}, \mu_{21}, \mu_{19}, \mu_{12}, \mu_9, \mu_8, \mu_3, \mu_2, \mu_{22}, \mu_{20}, \mu_{18}, \mu_{16}, \mu_{11}, \mu_4, \mu_1, \mu_{10}, \mu_{25}, \mu_{24}, \\ \mu_{17}, \mu_{15}, \mu_{13}, \mu_7, \mu_6, \mu_5, \mu_{14}.$$

$$P_{w_3}^* = \mu_{24}, \mu_{23}, \mu_{20}, \mu_{15}, \mu_9, \mu_6, \mu_3, \mu_{10}, \mu_{22}, \mu_{21}, \mu_{17}, \mu_{14}, \mu_{11}, \mu_7, \mu_1, \mu_2, \mu_{25}, \mu_{19}, \\ \mu_{18}, \mu_{16}, \mu_{13}, \mu_8, \mu_5, \mu_4, \mu_{12}.$$

As in previous examples, we consider a set of estimation functions that satisfies full admissibility. Under this assumption, it is not difficult to show that the *reduced problem* of this example is characterized by the following profile of preferences:

$$P_{f_1}^\varphi = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \emptyset, \{w_3\}.$$

$$P_{f_2}^\varphi = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \emptyset.$$

$$P_{w_1}^\varphi = f_2, f_1, w_1.$$

$$P_{w_2}^\varphi = f_2, f_1, w_2.$$

$$P_{w_3}^\varphi = f_1, f_2, w_3.$$

Note that the *reduced problem*  $(F, W, P^\varphi, q)$  of this example has no stable matchings, i.e.  $\mathcal{E}(F, W, P^\varphi, q) = \emptyset$ . However, it is also easy to show that the matching  $\mu_{11} = f_2, f_1, f_2$  is  $\varphi$ -stable.

In the following section, we introduce a restriction over agents' preferences called *bottom  $q$ -substitutability* that generalizes the condition of  *$q$ -substitutability* to matching problems with externalities and, under *full admissibility*, assures the existence of  $\varphi$ -stable matchings.

### 3.2 Bottom $q$ -substitutability

The condition of  *$q$ -substitutability* (Cantala, 2004) is a restriction on firms preferences that generalizes the well known condition of substitutability (Kelso and Crawford, 1982) to matching problems where firms have quotas of workers.<sup>5</sup> The condition of  *$q$ -substitutability* guarantees the existence of stable matchings in standard many-to-one matching problems with firms' quotas. According to Proposition 1 (and Corollary 1) the existence of  $\varphi$ -stable matchings can be established by imposing conditions on the matching problem that assure the existence of stable matchings in the *reduced problem*.

Before introducing our restriction on agents' preferences called *bottom  $q$ -substitutability*, we require some additional notation. For each firm  $f \in F$  and any subset of feasible workers  $S \in H_f$ , the matching  $\mu_{f,S}$  satisfies the following conditions: a)  $\mu_{f,S} \in A(f, S)$  and b)  $\mu' R_f^* \mu_{f,S}$  for all  $\mu' \in A(f, S)$ . In a similar way, for each worker  $w \in W$  and any agent  $a \in F \cup \{w\}$ , the matching  $\mu_{w,a}$  satisfies the following conditions: a)  $\mu_{w,a} \in A(w, a)$  and b)  $\mu' R_w^* \mu_{w,a}$  for all  $\mu' \in A(w, a)$ . Note that the matching  $\mu_{f,S}$  ( $\mu_{w,a}$ ) is the  $f$ 's ( $w$ 's)

---

<sup>5</sup>Let  $P_f$  be a preference relation of the firm  $f$ . The mapping  $Ch_f : 2^W \rightarrow H_f$  denotes the optimal choice of the firm  $f$ . For any subset of workers  $S \in 2^W$ , the  $f$ 's choice function satisfies the following conditions: a)  $Ch_f(S) \in H_f$  and b)  $Ch_f(S) R_f S'$  for all  $S' \subset S$ . Given any problem  $(F, W, P, q)$ , we say that  $f$ 's preferences are  *$q$ -substitutable* if for any subset of workers  $S \in 2^W$  such that  $w, w' \in S$  and  $w \neq w'$ , if  $w \in Ch_f(S)$  then  $w \in Ch_f(S \setminus \{w'\})$ .

least preferred matching where the firm  $f$  (the worker  $w$ ) and the subset of feasible workers  $S \in H_f$  (and the agent  $a \in F \cup \{w\}$ ) are matched. Note that these matchings are well defined, since agents' preferences are strict and complete over the set of feasible matchings. Let  $M_f = \{\mu \in \mathcal{M} : \mu = \mu_{f,S} \text{ and } S \in H_f\}$  denote the set of “least preferred matchings for the firm  $f$ ”. Given the previous notation, we are able to introduce the following definition.

**Definition 7** *For each firm  $f \in F$  and any subset of workers  $S \in 2^W$ , the mapping  $\Upsilon_f : 2^W \rightarrow \mathcal{M}$  satisfies the following conditions:*

1.  $\Upsilon_f(S) \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$ ; and
2.  $\Upsilon_f(S) R_f^* \mu'$  for all  $\mu' \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$ .

The mapping  $\Upsilon_f$  can be interpreted as the choice function of the firm  $f$  in the presence of externalities. This mapping is defined under a min-max argument where for any given subset of workers, firms choose the best matching among the worst possible assignments. Note that the choice function  $\Upsilon_f$  is well defined, since the preference relation  $P_f^*$  is strict and complete for each firm  $f \in F$  and the set of matchings  $\{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f \subset \mathcal{M}$  is always nonempty. For instance, it is clear that the matching  $\mu_{f,\emptyset}$  belongs to the set  $\{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$ , since by definition  $\mu_{f,\emptyset}(f) = \emptyset \subset S$  for any subset of workers  $S \subset W$ . It is also clear that  $\{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f = M_f$  whenever  $S = W$ , which implies that the choice function  $\Upsilon_f(W)$  maps the most preferred matching among the ones in the set  $M_f$ . This choice function allows us to introduce a notion of substitutability for matching problems with externalities.

**Definition 8** *We say that the preference profile  $P^*$  in the problem  $(F, W, P^*, q)$  satisfies the condition of bottom  $q$ -substitutability whenever for every firm  $f \in F$  and any set of workers  $S \in 2^W$  such that  $w, w' \in S$ , if  $w \in \mu(f)$  and  $\Upsilon_f(S) = \mu$  then  $w \in \mu'(f)$  and  $\Upsilon_f(S \setminus \{w'\}) = \mu'$ .*

Note that for each firm  $f$  and any feasible set of workers  $S \in H_f$ , there exists a unique matching in the set  $M_f$  that satisfies  $\mu(f) = S$ . Hence, under *full admissibility* the profile of

preferences of the *reduced problem* is fully characterized by the restricted preference relation over the set of least preferred matchings  $M_f$ . Thus, under *full admissibility* the profile of preferences  $P^*$  of the matching problem  $(F, W, P^*, q)$  is *bottom  $q$ -substitutable* if and only if the profile of preferences  $P^\varphi$  of the reduced problem  $(F, W, P^\varphi, q)$  is  *$q$ -substitutable*. Let  $\mathcal{BS}$  denote the set of all preference profiles that satisfy the condition of *bottom  $q$ -substitutability*. Then, we establish the following result.

**Theorem 2** *Let  $(F, W, P^*, q)$  be any matching problem with externalities. Suppose that  $P^* \in \mathcal{BS}$ , then under full admissibility the set of  $\varphi$ -stable matchings  $\mathcal{E}(F, W, P^*, q)$  is not empty.*

The previous result implies that the condition of *bottom  $q$ -substitutability* is sufficient but not necessary to assure the existence of  $\varphi$ -stable matchings. Consider, for instance, the following matching problem already introduced in Example 4. This matching problem induces a reduced problem with the following agents' preferences:

$$P_{f_1}^\varphi = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \emptyset, \{w_3\}.$$

$$P_{f_2}^\varphi = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \emptyset.$$

$$P_{w_1}^\varphi = f_2, f_1, w_1.$$

$$P_{w_2}^\varphi = f_2, f_1, w_2.$$

$$P_{w_3}^\varphi = f_1, f_2, w_3.$$

According to these preferences, the firm  $f_1$  should choose the set of workers  $\{w_1, w_3\}$  from the set  $\{w_1, w_3\}$ . However, when only the worker  $w_3$  is available, the firm  $f_1$  prefers to choose the empty set of workers. This simple example implies that the firm  $f_1$  is not willing to substitute the worker  $w_1$  with the worker  $w_3$ . Hence, this profile of preferences is not  *$q$ -substitutable*, which implies that the profile of preferences of the original problem  $P^*$  is not *bottom  $q$ -substitutable*. Furthermore, we already argued that the set of stable matchings of this reduced problem is empty. However, under *full admissibility* the matching  $\mu_{11} = f_2, f_1, f_2$  is  $\varphi$ -stable.

In the following example, we analyze a problem with *bottom  $q$ -substitutable* preferences that guarantees the existence of at least one  $\varphi$ -stable matching. In order to clarify all previous concepts and notation, we analyze the problem with some detail.

**Example 5** Consider a matching problem with two firms  $F = \{f_1, f_2\}$  with quotas  $q_{f_1} = q_{f_2} = 2$  and three workers  $W = \{w_1, w_2, w_3\}$ . The set of feasible matchings of this problem is given in the Table 3 of Example 2. Agents' preferences are given by the following lists of matchings:

$$P_{f_1}^* = \mu_6, \mu_4, \mu_5, \mu_8, \mu_1, \mu_7, \mu_2, \mu_{20}, \mu_{24}, \mu_{23}, \mu_{15}, \mu_{18}, \mu_{22}, \mu_{16}, \mu_{11}, \mu_{25}, \mu_{21}, \mu_{19}, \mu_{17}, \mu_{14}, \mu_{13}, \mu_{12}, \mu_3, \mu_{10}, \mu_9.$$

$$P_{f_2}^* = \mu_{14}, \mu_{13}, \mu_{21}, \mu_{16}, \mu_{12}, \mu_9, \mu_{15}, \mu_2, \mu_{17}, \mu_{19}, \mu_{22}, \mu_{23}, \mu_7, \mu_{10}, \mu_1, \mu_8, \mu_3, \mu_{25}, \mu_{24}, \mu_{20}, \mu_{18}, \mu_6, \mu_5, \mu_4, \mu_{11}.$$

$$P_{w_1}^* = \mu_1, \mu_2, \mu_3, \mu_5, \mu_6, \mu_7, \mu_8, \mu_4, \mu_9, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{10}, \mu_{17}, \mu_{18}, \mu_{19}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{24}, \mu_{25}, \mu_{20}.$$

$$P_{w_2}^* = \mu_2, \mu_8, \mu_9, \mu_{12}, \mu_{19}, \mu_{21}, \mu_{23}, \mu_3, \mu_1, \mu_4, \mu_{11}, \mu_{16}, \mu_{18}, \mu_{20}, \mu_{22}, \mu_{10}, \mu_5, \mu_6, \mu_7, \mu_{13}, \mu_{15}, \mu_{17}, \mu_{24}, \mu_{25}, \mu_{14}.$$

$$P_{w_3}^* = \mu_2, \mu_7, \mu_{11}, \mu_{14}, \mu_{17}, \mu_{21}, \mu_{22}, \mu_1, \mu_3, \mu_6, \mu_9, \mu_{15}, \mu_{20}, \mu_{23}, \mu_{24}, \mu_{10}, \mu_4, \mu_5, \mu_8, \mu_{13}, \mu_{16}, \mu_{18}, \mu_{19}, \mu_{25}, \mu_{12}.$$

As we argued before, in order to check if the condition of bottom  $q$ -substitutability is satisfied, it is enough to consider the restricted firms' preferences on the sets of matchings  $M_{f_1}$  and  $M_{f_2}$ :

$$\tilde{P}_{f_1}^* = \mu_1, \mu_2, \mu_{11}, \mu_{12}, \mu_3, \mu_{10}, \mu_9.$$

$$\tilde{P}_{f_2}^* = \mu_9, \mu_2, \mu_{10}, \mu_1, \mu_3, \mu_4, \mu_{11}.$$

It is easy to show that these preferences are bottom  $q$ -substitutable. Formally, we need to check the firms' choices for any possible subset of workers and establish the condition. For



instance, consider the case of the firm  $f_1$  and the set of workers  $S = \{w_1, w_2, w_3\}$ . According to  $f_1$ 's preferences, the following holds.

$$\begin{aligned}\Upsilon_{f_1}(S) &= \mu_1 \text{ with } \mu_1(f_1) = \{w_1, w_2\}; \\ \Upsilon_{f_1}(S \setminus \{w_1\}) &= \mu_{11} \text{ with } \mu_{11}(f_1) = \{w_2\}; \\ \Upsilon_{f_1}(S \setminus \{w_2\}) &= \mu_2 \text{ with } \mu_2(f_1) = \{w_1\} \text{ and} \\ \Upsilon_{f_1}(S \setminus \{w_3\}) &= \mu_1 \text{ with } \mu_1(f_1) = \{w_1, w_2\}.\end{aligned}$$

However, according to our previous arguments we can also check whether the reduced problem  $(F, W, P^\varphi, q)$  has  $q$ -substitutable preferences. It is easy to show that the reduced problem of this example has the following profile of preferences:

$$\begin{aligned}P_{f_1}^\varphi &= \{w_1, w_2\}, \{w_1\}, \{w_2\}, \emptyset, \{w_1, w_3\}, \{w_2, w_3\}, \{w_3\}. \\ P_{f_2}^\varphi &= \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_3\}, \{w_2\}, \emptyset, \{w_1, w_3\}. \\ P_{w_1}^\varphi &= f_1, f_2, w_1. \\ P_{w_2}^\varphi &= f_2, f_1, w_2. \\ P_{w_3}^\varphi &= f_2, f_1, w_3.\end{aligned}$$

It is clear that these preferences are  $q$ -substitutable, then a  $\varphi$ -stable matching exists. It is easy to show that the matching  $\mu_2$  is stable in the reduced problem  $(F, W, P^\varphi, q)$ . Hence, by the Corollary 1 the matching  $\mu_2$  is also  $\varphi$ -stable, i.e.  $\mu_2 \in \mathcal{E}(F, W, P^*, q)$ .

## 4 Pessimistic Agents

As we described before, *full admissibility* is a situation where every matching is considered admissible for every agent. This kind of estimation functions seems to be compatible with the absence of information about other agents' expectations, which leads agents to estimate that any matching can be considered admissible. In particular, note that under full admissibility agents are pessimistic in the sense that they consider admissible the subset of worst feasible matchings of the market. Sasaki and Toda (1996) show that this kind of pessimistic

expectations may be crucial to characterize the existence of stable assignments in matching problems with externalities. In particular, they notice that there are matching problems with externalities with no stable assignments when at least one of the agents is not pessimistic.

In the previous sections, we showed that there is at least one set of estimations that guarantees the existence of stable assignments under the domain of *bottom  $q$ -substitutable* preferences, i.e. the case of *full admissibility*. These kinds of estimations are exogenously determined. In general, arbitrary sets of pessimistic estimations do not guarantee to be admissible those matches which are candidates for being stable, even when preferences are *bottom  $q$ -substitutable*. In order to relax this condition, following a similar argument as in Hafalir (2008), we propose a set of estimations that rationalizes the expectations of pessimistic agents. Intuitively, in a setting where all agents are pessimistic, we consider that a matching cannot be admissible for an agent that knows there is a coalition which is able to block that matching. When all agent follow this simple argument, it is possible to characterize a set of estimations which guarantees that those matches that would be candidates for being stable will be also admissible for every agent.

Formally, we say that agents are *pessimistic* whenever the set of *estimation functions*  $\varphi$  satisfies the following conditions: 1) For each  $f \in F$ ,  $\mu_{f,S} \in \varphi_f(S)$  for all  $S \in H_f$  and 2) for all  $w \in W$ ,  $\mu_{w,a} \in \varphi_w(a)$  for all  $a \in F \cup w$ . Under the assumption that agents are *pessimistic*, it is possible to establish the following result.

**Proposition 2** *Assume that agents are pessimistic, then any stable matching of the reduced problem, i.e.  $\mu \in \mathcal{E}(F, W, P^\varphi, q)$ , is not blocked by any coalition or individual worker in the matching problem  $(F, W, P^*, q)$ .*

As we argue before, the existence of  $\varphi$ -stable matchings does not come from the previous result, since an arbitrary set of pessimistic estimations do not imply the existence of  $\varphi$ -admissible matchings.

## 4.1 Pessimistic Estimation Functions

In order to rationalize the estimation functions of pessimistic agents, we consider the following intuitive argument. Suppose that a firm  $f$  is planning to be matched with some feasible group of workers  $S \in H_f$ . When agents are *pessimistic*, the matching  $\mu_{f,S} \in A(f, S)$  is admissible by definition, i.e.  $\mu_{f,S} \in \varphi_f(S)$ . However, this firm  $f$  cannot consider admissible another matching  $\mu \in A(f, S) \setminus \{\mu_{f,S}\}$  that may be blocked by some coalition, i.e. a matching for which there exists a coalition  $\{f', S'\} \subset F \cup W \setminus \{f\}$  such that  $S' \in H_{f'}$ , whose preferences satisfy  $\mu_{f',S'} P_{f'}^* \mu$  and  $\mu_{w',f'} P_{w'}^* \mu$  for all  $w' \in S'$ . Intuitively, this firm  $f$  is able to anticipate that the matching  $\mu \in A(f, S)$  will be eventually blocked by the coalition  $\{f', S'\}$ . Formally, the set of pessimistic estimation functions is characterized by the following conditions.

**Definition 9** A matching  $\mu$  is admissible for the firm  $f$ , i.e.  $\mu \in \rho_f(\mu(f))$  if there is no coalition  $\{f', S'\} \subset F \cup W \setminus \{f\}$  such that  $S' \in H_{f'}$  that satisfies:

1.  $\mu_{f',S'} P_{f'}^* \mu$  and
2.  $\mu_{w',f'} P_{w'}^* \mu$  for all  $w' \in S'$ ;

and no subset of workers  $S'' \subset W$  that satisfies:

1.  $\mu_{w',w'} P_{w'}^* \mu$  for all  $w' \in S''$ .

In a similar way,

**Definition 10** A matching  $\mu$  is admissible for the worker  $w$ , i.e.  $\mu \in \rho_w(\mu(w))$  if there is no coalition  $\{f, S\} \subset F \cup W \setminus \{w\}$  that satisfies:

1.  $\mu_{f,S} P_f^* \mu$  and
2.  $\mu_{w,f} P_w^* \mu$  for all  $w \in S$ ;

and no subset of workers  $S'' \in W \setminus \{w\}$  such that:

1.  $\mu_{w',w'} P_{w'}^* \mu$  for all  $w' \in S''$ .

Let  $\rho = \{(\rho_f(\cdot), \rho_w(\cdot)) : f \in F, w \in W \text{ and } P^*\}$  denote the set of *pessimistic estimation functions*. Note that these *estimations* depend on agents' preferences and rationalize the conjecture that all agents are *pessimistic*. In general, the set of *pessimistic estimation functions*  $\rho$  does not satisfy the condition of *full admissibility*, as we show in the following example.

**Example 6** Consider the matching problem with externalities already introduced in the Example 5. The set of least preferred feasible matchings of the firms  $f_1$  and  $f_2$  are:

$$M_{f_1} = \{\mu_1, \mu_2, \mu_{11}, \mu_{12}, \mu_3, \mu_{10}, \mu_9\}$$

$$M_{f_2} = \{\mu_9, \mu_2, \mu_{10}, \mu_1, \mu_3, \mu_4, \mu_{11}\}$$

Consider the following set of feasible matchings  $A(\{f_1, \{w_1, w_2\}\}) = \{\mu_1, \mu_4\}$ . By assumption the matching  $\mu_1$  is considered admissible by the firm  $f_1$ , i.e.  $\mu_1 \in \rho_{f_1}(\{w_1, w_2\})$ . However, the matching  $\mu_4 = f_1, f_1, w_3$  can be blocked by the coalition  $\{f_2, w_3\} \subset F \cup W \setminus \{f_1\}$ , since  $\mu_{f_2, \{w_3\}} = \mu_1$  and  $\mu_{w_3, \{f_2\}} = \mu_1$  and according to  $f_2$  and  $w_3$  preferences,

$$1. \mu_1 P_{f_2}^* \mu_4 \text{ and}$$

$$2. \mu_1 P_{w_3}^* \mu_4.$$

Hence, the matching  $\mu_4$  cannot be admissible for the firm  $f_1$ , i.e.  $\mu_4 \notin \rho_{f_1}(\{w_1, w_2\})$ . Similarly, consider the set of feasible matchings  $A(w_1, f_1) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8\}$ . As in the previous case, by assumption the matching  $\mu_4$  is admissible for the firm  $f_1$ , i.e.  $\mu_4 \in \rho_{w_1}(f_1)$ . However, the matching  $\mu_5 = f_1, w_2, w_3$  cannot be admissible, since the coalition  $\{f_2, \{w_2, w_3\}\} \subset F \cup W \setminus \{f_1\}$  has incentives to block it. It is easy to show that,  $\mu_{f_2, \{w_2, w_3\}} = \mu_2$ ,  $\mu_{w_2, \{f_2\}} = \mu_3$  and  $\mu_{w_3, \{f_2\}} = \mu_1$ . And according to  $f_2$ ,  $w_2$  and  $w_3$  preferences, the following is satisfied,

$$1. \mu_2 P_{f_2}^* \mu_5,$$

$$2. \mu_3 P_{w_2}^* \mu_5 \text{ and}$$

$$3. \mu_1 P_{w_3}^* \mu_5.$$

Thus  $\mu_5 \notin \rho_{w_1}(f_1)$ .

The set of pessimistic estimation functions of this problem is given by the following sets of admissible matchings for each agent.

**For the firm  $f_1$ :**

$$\begin{aligned}\rho_{f_1}(\{w_1, w_2\}) &= \{\mu_1\}; \\ \rho_{f_1}(\{w_1, w_3\}) &= \{\mu_3\}; \\ \rho_{f_1}(\{w_2, w_3\}) &= \{\mu_{10}\}; \\ \rho_{f_1}(\{w_1\}) &= \{\mu_2, \mu_7\}; \\ \rho_{f_1}(\{w_2\}) &= \{\mu_{11}, \mu_{16}, \mu_{22}\}; \\ \rho_{f_1}(\{w_3\}) &= \{\mu_9, \mu_{15}, \mu_{23}\} \text{ and} \\ \rho_{f_1}(\phi) &= \{\mu_{12}, \mu_{13}, \mu_{14}, \mu_{17}, \mu_{19}, \mu_{21}\}.\end{aligned}$$

**For the firm  $f_2$ :**

$$\begin{aligned}\rho_{f_2}(\{w_1, w_2\}) &= \{\mu_9, \mu_{12}\}; \\ \rho_{f_2}(\{w_1, w_3\}) &= \{\mu_{11}\}; \\ \rho_{f_2}(\{w_2, w_3\}) &= \{\mu_2\}; \\ \rho_{f_2}(\{w_1\}) &= \{\mu_{10}, \mu_{15}, \mu_{16}\}; \\ \rho_{f_2}(\{w_2\}) &= \{\mu_3, \mu_8\}; \\ \rho_{f_2}(\{w_3\}) &= \{\mu_1, \mu_7\} \text{ and} \\ \rho_{f_2}(\phi) &= \{\mu_4, \mu_5, \mu_6, \mu_{18}, \mu_{20}, \mu_{24}\}.\end{aligned}$$

**For the worker  $w_1$ :**

$$\begin{aligned}\rho_{w_1}(f_1) &= \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_7\}; \\ \rho_{w_1}(f_2) &= \{\mu_{10}, \mu_{11}, \mu_{12}, \mu_{15}, \mu_{16}\}; \text{ and} \\ \rho_{w_1}(w_1) &= \{\mu_{19}, \mu_{20}, \mu_{21}, \mu_{22}\}.\end{aligned}$$

**For the worker  $w_2$ :**

$$\rho_{w_2}(f_1) = \{\mu_1, \mu_{10}, \mu_{16}, \mu_{22}\};$$

$$\rho_{w_2}(f_2) = \{\mu_2, \mu_3, \mu_8, \mu_{12}\} \text{ and}$$

$$\rho_{w_2}(w_2) = \{\mu_7, \mu_{13}, \mu_{14}\}.$$

**For the worker  $w_3$ :**

$$\rho_{w_3}(f_1) = \{\mu_3, \mu_9, \mu_{10}, \mu_{15}, \mu_{23}\};$$

$$\rho_{w_3}(f_2) = \{\mu_1, \mu_2, \mu_7, \mu_{11}\} \text{ and}$$

$$\rho_{w_3}(w_3) = \{\mu_8, \mu_{12}, \mu_{19}\}.$$

*Note that according to these estimations neither of the following matchings is  $\rho$ -admissible:*

$$\{\mu_4, \mu_5, \mu_6, \mu_8, \mu_9, \mu_{11}, \mu_{13}, \mu_{14}, \mu_{15}, \mu_{17}, \mu_{16}, \mu_{18}, \mu_{19}, \mu_{20}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{24}, \mu_{25}\}.$$

As was already shown in the Example 5, the matching  $\mu_2 = f_1, f_2, f_2$  must be stable in the *reduced problem*  $(F, W, P^\rho, q)$  associated with this example. According to Proposition 2, the matching  $\mu_2 = f_1, f_2, f_2$  cannot be blocked by any worker or coalition in the original matching problem with externalities  $(F, W, P^*, q)$ , since all agents are pessimistic. Furthermore, note also that the matching  $\mu_2$  is  $\rho$ -admissible, since  $\mu_2 \in \rho_a(\mu_2(a))$  for all  $a \in F \cup W$ . This last observation implies that the matching  $\mu_2$  is a  $\rho$ -stable matching, i.e.  $\mu_2 \in \mathcal{E}_\rho(F, W, P^*, q)$ . The following result shows that this characteristic holds in general. In particular, we show that given the set of *pessimistic estimation functions*  $\rho$ , the condition of *bottom  $q$ -substitutability* is sufficient to assure the existence of  $\rho$ -stable matchings.

**Theorem 3** *Given the set of pessimistic estimation functions  $\rho$ , if  $P^* \in \mathcal{BS}$  then the set of  $\rho$ -stable matchings,  $\mathcal{E}_\rho(F, W, P^*, q)$  is not empty.*

According to the previous result, the set of pessimistic estimation functions not only rationalizes the set of matchings that pessimistic agents consider admissible but also guarantees that every stable matching of the reduced problem will be admissible for all agents. In contrast to the case of *full admissibility*, not all matchings are required to be admissible in order to characterize the existence of  $\varphi$ -stable matchings; hence the previous example is also sufficient to establish the following result.

**Proposition 3** *The condition of full admissibility is neither necessary nor sufficient to assure the existence of  $\varphi$ -stable matchings in many-to-one matching problems with externalities.*

## 5 The Core in Matching Problems with Externalities

The core is the set of matchings not blocked by any coalition (Shapley and Shubik, 1972, Echenique and Oviedo, 2004). One of the main results of the two-sided matching literature is the equivalence between the set of pairwise stable matchings and the core in the marriage problem (Roth and Sotomayor, 1990). This result is important because it shows that the analysis of a simple type of deviations is sufficient to characterize the existence of stable matchings under more complex kinds of deviations such as coalitional deviations considered in the analysis of the core. Hence, this equivalence between both sets of matchings highlights the generality of the set of stable matchings as a solution concept for this kind of cooperative games. Sasaki and Toda (1996) introduce a notion of the core for marriage problems with externalities. According to their notion of the core, once externalities are considered not only do the set of  $\varphi$ -stable and the core not coincide but also the core may be empty. This results contrasts with the case of standard marriage problem where, as we argued before, the core and the set of stable matchings always coincide. In this regard, other authors have also analyzed the relationship between the core and the set of stable matching in problems with externalities, reaching similar results (See for instance, Hafalir (2008) and Muncu and Saglam (2010)).

Sasaki and Toda (1996)'s core may be empty, since in general deviating agents do not take into account the set of matchings that they consider admissible. Under this notion of the core, members of coalitions deviate even when they could be worse off under admissible matchings after deviating. We propose an alternative notion of the core by assuming that agents always consider their *estimation functions* when they plan to deviate as members of any coalition. Formally,

**Definition 11** *A coalition  $A \subset F \cup W$  blocks the matching  $\mu$ , whenever there is another matching  $\hat{\mu} \neq \mu$  such that:*

1.  $\hat{\mu}(a) \subset A$  for all  $a \in A$ ;
2.  $\mu' P_f^* \mu$  for all  $\mu' \in \varphi_f(\hat{\mu}(f))$  and all  $f \in A$ ; and
3.  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w(\hat{\mu}(w))$  and all  $w \in A$ .

The core for a given the set of *estimation functions*  $\varphi$ , or simply the  $\varphi$ -core, is the set of  $\varphi$ -admissible matchings not blocked by any coalition  $A \subset F \cup W$ . Let  $\mathcal{C}_\varphi(F, W, P^*, q)$  denote the  $\varphi$ -core. The following result provides an interesting property of the  $\varphi$ -core.

**Theorem 4** *Let  $(F, W, P^*, q)$  be any matching problem with externalities and let  $\varphi$  be any set of estimation functions, then  $\mathcal{E}_\varphi(F, W, P^*, q) = \mathcal{C}_\varphi(F, W, P^*, q)$ .*

The previous result has some crucial implications about the existence and properties of the core for matching problems with externalities. First of all, it is clear that all existence results and properties of the set of  $\varphi$ -stable matchings extend to the  $\varphi$ -core. Secondly, the equivalence between the set of  $\varphi$ -stable matchings and the  $\varphi$ -core does not depend on the set of estimation functions.

## 6 Conclusions

In this paper, we analyze the existence of stable matchings in matching problems with externalities. We argue that standard results about the existence of stable matchings cannot be trivially extended to this setting. First, agents' preferences should be defined over the set of all feasible matchings of the problem instead of the set of agents on the opposite side of the market. Once externalities are considered, agents should anticipate the reaction of the rest of agents in the face of potential deviations.

We extend the notion of stability proposed by Sasaki and Toda (1996) based on the concept of *estimation functions*. The set of *estimation functions* represents a belief about the set of matchings that agents consider admissible given a conjecture about the reactive behaviors of agents. In general, the set of *estimation functions* is not uniquely defined and may be either exogenously or endogenously determined.



We show that in many-to-one problems, a  $\varphi$ -stable matchings may not exist. Furthermore, it is possible to find instances of the problem with no  $\varphi$ -stable matchings for any given set of *estimations*. This impossibility theorem contrasts with previous results in marriage problems with externalities, where there exists at least one set of *estimations* that assures the existence of  $\varphi$ -stable matchings. Given this impossibility result, we focus on the case of *full admissibility*. In this case, we provide a restriction on firms' preferences called *bottom  $q$ -substitutability* that guarantees the existence of at least one  $\varphi$ -stable matching.

Under the assumption that agents are *pessimistic*, it is possible to construct a set of *pessimistic estimations* that rationalizes the set of estimation functions of pessimistic agents. The set of *pessimistic estimations* depends on agents' preferences and guarantees the existence of  $\varphi$ -stable matchings, providing preferences are *bottom  $q$ -substitutable*. In addition, our results show that the assumption of *full admissibility* is neither necessary nor sufficient for assuring the existence of  $\varphi$ -stable matchings.

Finally, we analyze the relationship between a notion that we propose of the core and the set of stable matchings in problems with external effects. The notion of the core analyzed in previous literature may be empty for some instances of the matching problem with externalities. We propose an alternative notion of the core where agents consider their estimation functions called  $\varphi$ -core. We show that the set of  $\varphi$ -stable matchings and the  $\varphi$ -core always coincide for any given set of estimations  $\varphi$ . As a final remark, it is clear that all results presented in this paper extend to the marriage problem with externalities, since this is a particular case of the many-to-one matching problem with externalities.

## References

- [1] Azrieli, Y. (2009) "On pure conjectural equilibrium with non-manipulable information" *International Journal of Game Theory* **38**, 209-219.
- [2] Bando, K. (2012) "Many-to-one matching markets with externalities of firms", *Journal of Mathematical Economics*, Volume 48, Issue 1, 14-20.

- [3] Cantala, D. (2004) "Restabilizing matching markets at senior level" *Games and Economic Behavior* **48**, 1-17.
- [4] Dutta, B. and Masso, J. (1997) "Stability of marriage when individuals have preferences over colleagues" *Journal of Economic Theory* **75**, 464-475.
- [5] Echenique, F. and Oviedo, J. (2004) "Core many-to-one matchings by fixed-point methods" *Journal of Economic Theory* **115**, 358-376.
- [6] Echenique, F. and Yenmez, M. B. (2007) "A solution to matching with preferences over colleagues" *Games and Economic Behavior* **59**, 46-71.
- [7] Gale, D. and Shapley, L. (1962) "College admission and the stability of marriage" *American Mathematical Monthly* **69**, 9-15.
- [8] Hafalir, I. E. (2008) "Stability of marriage with externalities" *International Journal of Game Theory* **37**, 353-369.
- [9] Kelso, A. and Crawford, V. (1982) "Job matching, coalition formation and gross substitutes" *Econometrica* **50**, No. 6, 1483-1504.
- [10] Klaus, B. and Klijn, F. (2005) "Stable matchings and preferences of couples" *Journal of Economic Theory* **121**, 75-106.
- [11] Kurucu, G. (2007) "Negative network externalities in two-sided markets: a competition approach" MPRA Paper No. **9746**, 1-24.
- [12] Li, S. (1993) "Competitive matching equilibrium and multiple principal-agent models" Mimeo, University of Minnesota.
- [13] Mumcu, A. and Saglam, I. (2006) "One to one matching with interdependent preferences" MPRA Paper No. **1908**, 1-28.
- [14] Mumcu, A. and Saglam, I. (2007) "The core of housing market with externalities" *Economics Bulletin*, vol. 3, No. **55**, 1-5.

- [15] Mumcu, A. and Saglam, I. (2010) “Stable one-to-one matchings with externalities” *Mathematical Social Sciences* **60**, 154-159.
- [16] Roth, A.E. and Sotomayor, M. (1990) “Two sided matching: A study in game theoretic modeling and analysis” Cambridge University Press, 1-265.
- [17] Roy, P. (2004) “Marriage markets with externalities” Indian Statistical Institute, Delhi, 1-9.
- [18] Rubinstein, A. and Wolinsky, A. (1994) “Rationalizable conjectural equilibrium: Between Nash and rationalizability” *Games and Economic Behavior* **6**, 299-311.
- [19] Sasaki, H. and Toda, M. (1996) “Two sided matching problems with externalities” *Journal of Economic Theory* **70**, 93-108.
- [20] Shapley, L. and Shubik, M. (1969) “On the core of an economic system with externalities” *The American Economic Review* **59**, 678-684.
- [21] Shapley, L. and Shubik, M. (1972) “The assignment game I: The core” *International Journal of Game Theory* **1**, 111-130.

## 7 Appendix: Proofs

### *Proof of Lemma 1:*

**Proof.** Suppose that  $\mathcal{E}(F, W, P^*, q) = \emptyset$  while  $\mathcal{E}_{\varphi^*}(F, W, P^*, q) \neq \emptyset$  for an arbitrary set of estimations  $\varphi^*$ . Consider any matching such that  $\mu \in \mathcal{E}_{\varphi^*}(F, W, P^*, q)$ , this matching is  $\varphi^*$ -admissible and not blocked by any worker or coalition given  $\varphi^*$ . Since  $\mathcal{E}(F, W, P^*, q) = \emptyset$ , we have two cases under *full admissibility*:

Case 1: The matching  $\mu$  is blocked by at least one coalition  $\{f, S\}$  such that  $S \in H_f$ . Hence, it is satisfied 1.  $\mu' P_f^* \mu$  for all  $\mu' \in A(f, S)$  and 2.  $\mu'' P_w^* \mu$  for all  $\mu'' \in A(w, f)$  for all

$w \in S$ . By definition,  $\varphi_f^*(S) \subset A(f, S)$  and  $\varphi_w^*(f) \subset A(w, f)$  for all  $f$  and  $w$  in  $F \cup W$ . Then it is clear that a)  $\mu' P_f^* \mu$  for all  $\mu' \in \varphi_f^*(S)$  and b)  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w^*(f)$  and all  $w \in S$ , which is a contradiction.

Case 2: The matching  $\mu$  is blocked by some worker  $w \in W$ . Then  $\mu(w) \neq w$  and  $\mu' P_w^* \mu$  for all  $\mu' \in A(w, w)$ , this implies that  $\mu' P_w^* \mu$  for all  $\mu' \in \varphi_w^*(w)$ , which is a contradiction. This completes the proof. ■

### ***Proof of Proposition 1:***

**Proof.** Assume that the matching  $\mu$  is blocked by some coalition  $\{f, S\}$  such that  $S \in H_f$  in the problem  $(F, W, P^*, q)$ , but stable in the reduced problem, i.e.  $\mu \in \mathcal{E}(F, W, P^\varphi, q)$ . If  $\mu' P_f^* \mu$  for all  $\mu' \in \varphi_f^*(S)$  implies that  $\mu_f^S P_f^* \mu$ , since  $\mu \in \varphi_a(\mu(a))$  for all  $a \in F \cup W$ . We know that  $\mu R_f^* \mu_f^{\mu(f)}$ ; hence  $\mu_f^S P_f^* \mu_f^{\mu(f)}$  which implies  $S P_f^\varphi \mu(f)$ . Assume that there is some worker such that  $w \in \mu(f) \cap S$ , this implies that  $\mu(w) = \{f\}$  and by assumption  $\mu \in \varphi_w(\mu(w))$ . Since  $\mu$  is blocked by  $\{f, S\}$ , we know that  $\mu'' P_{w'}^* \mu$  for all  $\mu'' \in \varphi_{w'}^*(f)$  and all  $w' \in S$ . It is impossible that  $\mu \in \varphi_w(f)$ , hence  $\mu(f) \cap S = \emptyset$ . If  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w(f)$  implies that  $\mu_w^f P_w^* \mu$ , since  $\mu \in \varphi_a(\mu(a))$  for all  $a \in F \cup W$ . We know that  $\mu R_w^* \mu_w^{\mu(w)}$ , hence  $\mu_w^f P_w^* \mu_w^{\mu(w)}$  which implies that  $f P_w^\varphi \mu(w)$  for all  $w \in S$ . These conditions imply that  $\mu(f) \neq S$ , a)  $S P_f^\varphi \mu(f)$  and b)  $f P_w^\varphi \mu(w)$  for all  $w \in S$ , which is a contradiction.

Now assume that  $\mu$  is blocked by an individual worker  $w \in W$ . In this case, we have that  $\mu(w) \neq w$  and  $\mu' P_w^* \mu$  for all  $\mu' \in \varphi_w(w)$ . By a similar argument as before, this implies that  $\mu_w^w P_w^* \mu_w^{\mu(w)}$ ; hence  $w P_w^\varphi \mu(w)$ , which is a contradiction.

Since the matching  $\mu \in \mathcal{E}(F, W, P^\varphi, q)$  is  $\varphi$ -admissible and not blocked by any worker or coalition; hence, we know that  $\mu$  is  $\varphi$ -stable. This completes the proof. ■

### ***Proof of Theorem 2:***

**Proof.** Assume that the condition of *full admissibility* holds. For each firm  $f \in F$ , we define the  $f$ 's choice function as  $Ch_f(S) = \mu(f)$  such that  $\Upsilon_f(S) = \mu$  for any  $S \subset W$ . First, we have to show that for each  $f \in F$  and any  $S \subset W$  the choice function  $Ch_f$  maps the best subset of workers in  $S$  according to the preference relation  $P_f^\varphi$ . We know that for any  $S \subset W$ ,

it is satisfied that  $\Upsilon_f(S) \in M_f$  and  $\Upsilon_f(S) R_f^* \mu'$  for all  $\mu' \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$ , then by definition  $\mu(f) \subset S$  whenever  $\Upsilon_f(S) = \mu$ . Assume that there is some subset of workers  $S' \subset S$ , such that  $S' P_f^\varphi \mu(f)$  for  $\Upsilon_f(S) = \mu$ . Hence, the preference relation  $S' P_f^\varphi \mu(f)$  implies that  $\mu_f^{S'} P_f^* \mu_f^{\mu(f)}$ . By the condition of *full admissibility*  $\Upsilon_f(S) = \mu = \mu_f^{\mu(f)}$ , since  $\mu_f^{S'} = \mu_{f,S'} \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$ , which is a contradiction. Hence,  $\mu(f) R_f^\varphi S'$  for all  $S' \subset S$  where  $\Upsilon_f(S) = \mu$ . This implies that the mapping  $Ch_f(S) = \mu(f)$  such that  $\Upsilon_f(S) = \mu$  for any  $S \subset W$  is a well defined choice function for each firm  $f$ .

Now we have to show that the preference profile  $P^\varphi$  satisfies the condition of *q-substitutability*. Suppose that  $w, w' \in S$  and  $w \in \mu(f)$  where  $\Upsilon_f(S) = \mu$ , this implies that  $w \in Ch_f(S)$ . By *bottom q-substitutability*, we know that  $w \in \mu'(f)$  where  $\Upsilon_f(S \setminus \{w'\}) = \mu'$ , this implies that  $w \in Ch_f(S \setminus \{w'\})$ , then the preferences profile  $P^\varphi$  satisfies *q-substitutability*. Hence, the *reduced problem*  $(F, W, P^\varphi, q)$  has at least one stable matching. Since by *full admissibility* any feasible matching of the problem is  $\varphi$ -admissible, then  $\mathcal{E}(F, W, P^*, q)$  is not empty. This completes the proof. ■

### ***Proof of Proposition 2:***

**Proof.** Assume that the matching  $\mu$  is stable in the *reduced problem*  $(F, W, P^\varphi, q)$  but blocked by some coalition  $\{f, S\}$  such that  $S \in H_f$  in the problem  $(F, W, P^*, q)$ . Hence,  $\mu' P_f^* \mu$  for all  $\mu' \in \varphi_f(S)$  implies that  $\mu_f^S P_f^* \mu$  while  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w(f)$ , which implies that  $\mu_w^f P_w^* \mu$ . By assumption,  $\mu_{f,\mu(f)} = \mu_f^{\mu(f)} \in \varphi_f(\mu(f))$  and  $\mu_{w,\mu(w)} = \mu_w^{\mu(w)} \in \varphi_w(\mu(w))$ . Hence,  $\mu R_f^* \mu_f^{\mu(f)}$  and  $\mu R_w^* \mu_w^{\mu(w)}$  even if the matching  $\mu$  is not  $\varphi$ -admissible. Then  $\mu_f^S P_f^* \mu_f^{\mu(f)}$  and  $\mu_w^f P_w^* \mu_w^{\mu(w)}$  for all  $w \in S$ , which imply that a)  $S P_f^\varphi \mu(f)$  and b)  $f P_w^\varphi \mu(w)$  for all  $w \in S$ , which is a contradiction.

Now suppose that  $\mu$  is blocked by a worker  $w \in W$ . In a similar way, we have that  $\mu(w) \neq w$  and  $\mu' P_w^* \mu$  for all  $\mu' \in \varphi_w(w)$  which implies  $\mu_w^w P_w^* \mu_w^{\mu(w)}$ . Hence,  $w P_w^\varphi \mu(w)$ , which is a contradiction. This completes the proof. ■

### ***Proof of Theorem 3:***

**Proof.** Let  $(F, W, P^\rho, q)$  be the *reduced problem* associated with  $(F, W, P^*, q)$ . By the con-

dition of *bottom  $q$ -substitutability*, there exists at least one stable matching in the *reduced problem*, say  $\mu^* \in \mathcal{E}(F, W, P^\rho, q)$ .

We have to show that  $\mu^*$  is  $\rho$ -admissible. Suppose in contradiction that  $\mu^* \notin \rho_f(\mu^*(f))$  for some firm  $f \in F$ . There are two cases:

**Case 1:** There exists a coalition  $\{f', S'\} \subset F \cup W \setminus \{f\}$  such that  $S' \in H_{f'}$ ,  $\mu_{f', S'} P_{f'}^* \mu$  and  $\mu_{w', f'} P_{w'}^* \mu$  for all  $w' \in S'$ . Since agents are pessimistic,  $\mu_{f', S'} = \mu_{f'}^{S'} \in \rho_{f'}(S')$  and  $\mu_{w', f'} = \mu_{w'}^{f'} \in \rho_{w'}(f')$ , then  $\mu_{f'}^{S'} P_{f'}^* \mu_{f'}^{\mu^*(f')}$  and  $\mu_{w'}^{f'} P_{w'}^* \mu_{w'}^{\mu^*(w')}$  for all  $w' \in S'$ . This implies that  $S' P_{f'}^\rho \mu^*(f')$  and  $f' R_{w'}^\rho \mu^*(w')$  for all  $w' \in S'$ , which is a contradiction.

**Case 2:** There exists a subset of workers  $S'' \subset W$  such that  $\mu_{w', w'} P_{w'}^* \mu$  for all  $w' \in S''$ . In a similar way as before, we know that  $\mu_{w', w'} = \mu_{w'}^{w'} \in \rho_{w'}(w')$ . Hence,  $\mu_{w'}^{w'} P_{w'}^* \mu_{w'}^{\mu^*(w')}$  implies that  $w' P_{w'}^\rho \mu^*(w')$  for all  $w' \in S''$ , which is a contradiction.

Given that  $f$  was any arbitrary firm, this implies that  $\mu^* \in \rho_f(\mu^*(f))$  for all  $f \in F$ . A similar argument applies for any worker. Hence, the matching  $\mu^*$  is  $\rho$ -admissible, i.e.  $\mu^* \in \rho_a(\mu^*(a))$  for all  $a \in F \cup W$ . Then, the matching  $\mu^*$  is  $\rho$ -stable. This completes the proof. ■

#### ***Proof of Theorem 4:***

**Proof.** Suppose that the matching  $\mu \in \mathcal{E}_\varphi(F, W, P^*, q)$  but  $\mu \notin \mathcal{C}_\varphi(F, W, P^*, q)$ , then there is another matching  $\hat{\mu} \neq \mu$  and one coalition  $A \subset F \cup W$  which blocks the matching  $\mu$ . Take any firm  $f \in A$  and the subset of workers such that  $\hat{\mu}(f) \subset A$ , obviously  $\hat{\mu}(w) = \{f\} \subset A$  for all  $w \in \hat{\mu}(f)$ . It is satisfied that: 1)  $\mu' P_f^* \mu$  for all  $\mu' \in \varphi_f(\hat{\mu}(f))$  and 2)  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w(f)$  and all  $w \in \hat{\mu}(f)$ . Then the coalition  $\{f, \hat{\mu}(f)\}$  blocks the matching  $\mu$ . If there is no firm in the coalition  $A$ , take any worker  $w \in A$ , obviously  $\hat{\mu}(w) = \{w\} \subset A$  and it is satisfied that: 1)  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w(w)$ . Hence, any individual worker  $w \in A$  blocks the matching  $\mu$ , which is a contradiction.

On the other hand, suppose that  $\mu \in \mathcal{C}_\varphi(F, W, P^*, q)$  but  $\mu \notin \mathcal{E}_\varphi(F, W, P^*, q)$ , then there is at least a coalition,  $\{f, S\}$ , or an individual worker,  $w \in W$ , which blocks the matching  $\mu$ . Set the matching  $\hat{\mu}$ , such that  $\hat{\mu}(f) = S$ , obviously  $\hat{\mu} \neq \mu$ , and  $A = \{f, S\}$ . By definition, it is satisfied that: 1)  $\mu' P_f^* \mu$  for all  $\mu' \in \varphi_f(\hat{\mu}(f))$  and all  $f \in A$  and 2)  $\mu'' P_w^* \mu$  for all

$\mu'' \in \varphi_w(\hat{\mu}(w))$  and all  $w \in A$ , then  $\mu \notin \mathcal{C}_\varphi(F, W, P^*, q)$ . Suppose that an individual worker blocks the matching  $\mu$ , set  $A = \{w\}$  and  $\hat{\mu}(w) = w$ , obviously  $\hat{\mu} \neq \mu$  and it is satisfied that: 1)  $\mu'' P_w^* \mu$  for all  $\mu'' \in \varphi_w(\hat{\mu}(w))$  and all  $w \in A$ , which is a contradiction. This completes the proof. ■